ON SUBSPACES OF BANACH SPACES WITHOUT QUASICOMPLEMENTS

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ABSTRACT
It is proved that for uncountable Γ, $c_0(\Gamma)$ has no quasicomplement in $m(\Gamma)$.

Let X be a Banach space and let Y be a closed subspace of X. A closed subspace Z of X is said to be a quasicomplement of Y in X if $Z \cap Y = \{0\}$ and $Z + Y$ is dense in X. This notion was introduced by Murray [10] who proved that in a separable reflexive Bananch space X every closed subspace has a quasicomplement. This result was generalized by Mackey [9] who proved that the same is true for every separable space X. Another generalization of Murray's result was obtained in [6] where it was proved that also in non-separable reflexive spaces every closed subspace has a quasicomplement. It was observed in [8] that by combining the argument of [6] with that of [1] the following result can be proved.

THEOREM 1. *Let X be a weakly compactly generated* (WCG) *Banach space. Let Y be a closed subspace of X which is also* WCG. *Then Y has a quasicomplement in X,*

A Banach space X is called WCG if there is a weakly compact set S in X such that X is the closed linear span of S. Since every separable or reflexive space is WCG, Theorem 1 includes the results of [6], [9] and [10] mentioned above. The natural question whether every closed subspace of an arbitrary Banach space has a quasicomplement was left open in all the preceding papers. It is the purpose of this note to prove that the answer to this question is negative. Our approach was strongly motivated by the paper [2] of Bade where the question of the existence of a quasicomplement to c_0 in m was discussed.

Let Γ be an abstract set, let $m(\Gamma)$ denote the space of bounded real-valued functions on Γ with the supremum norm. Let $c_0(\Gamma)$ denote the subspace of $m(\Gamma)$

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consisting of those f which vanish at infinity i.e., for which $\{\gamma; |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$. It is easily verified (cf. e.g. [8]) that $c_0(\Gamma)$ is WCG for every Γ while $m(\Gamma)$ is not WCG if Γ is infinite. As indicated by the abstract, we shall prove here that for uncountable Γ, $c_0(\Gamma)$ has no quasicomplement in $m(\Gamma)$. This fact is a special case of

THEOREM 2. *Let K be a compact Hausdorff F space and let $X = C(K)$ be the Banach space of continuous real valued functions on K. Let Y be a* WCG *subspace of X which contains a subspace isomorphic to $c_0(\Gamma)$ for some uncountable Γ. Then Y has no quasicomplement in X.*

A topological space is called an F space if every two disjoint open F_σ sets have disjoint closures. A topological space is called extremally disconnected if the closure of every open set is open (cf. [4] for a study of those spaces). Since $m(\Gamma) = C(\beta\Gamma)$ where $\beta\Gamma$ is the Stone Cech compactification of Γ, and since $\beta\Gamma$ is extremally disconnected and thus also an F space, it is clear that Theorem 2 contains the statement appearing in the abstract.

Proof of Theorem 2. Assume that there is a quasicomplement Z to Y in X. Let $T: X \to X/Z$ be the quotient map. By the definition of quasicomplement the image TY of Y is dense in X/Z. Since Y is WCG it follows that also X/Z is WCG.

We now recall the fact (cf. [1]) that if U is a WCG Banach space then the unit cell of U^* is w^* sequentially compact. Indeed if S is a w compact subset of U which generates U and if $\phi: U^* \to C(S)$ is the map defined by $\phi u^*(s)_{\iota} = u^*(s)$, then ϕ maps homeomorphically the unit cell of U^* in its w^* topology into a w compact subset of $C(S)$ which is sequentially compact by the Eberlein-Smulian theorem. Another fact we recall is that if K is a compact Hausdorff F space then in $C(K)^*$, w^* convergence of a sequence implies its convergence in the w topology. This fact is essentially due to Grothendieck [5] who proved it for extremally disconnected K. (For the extension of this result to general F spaces cf. [7] [11], [12].) From the preceding two facts it follows in particular that every bounded operator from a $C(K)$ space with K a compact Hausdorff F space into a WCG space is weakly compact. In particular the quotient map $T: X \to X/Z$ is weakly compact.

Let V be a subspace of Y which is isomorphic to $c_0(\Gamma)$ with Γ uncountable, and let T_0 be the restriction of T to V. Since T is w compact the same is true for T_0 and thus also for T_0^*: $(X/Z)^* \to V^*$. Since V^* is isomorphic to $l_1(\Gamma)$ and in $l_1(\Gamma)$ every w compact set is norm compact (cf. [3, p. 33]) it follow that T_0^* is a norm compact operator and thus has a separable range. On the other hand, by the definition of a quasicomplement the operator T_0 is one-to-one and hence its adjoint has a w^* dense range. However since Γ is uncountable there is no separable w^* dense subspace of $l_1(\Gamma) = c_0(\Gamma)^*$ and this concludes the proof.

Added in proof: Haskell P. Rosenthal has recently proved the existence of quasicomplements in some general situations which are not covered by Theorem 1 above. In particular he proved that c_0 has a quasicomplement in m.

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